A pyramid scheme is a business model in which payment is made primarily for enrolling other people into the scheme. Some schemes involve a legitimate business venture, but in others no product or services are delivered. A typical pyramid scheme combines a plausible business opportunity (such as a dealership) with a recruiting operation that promises substantial rewards. A recruited individual makes an initial payment, and can earn money by recruiting others who also make a payment; the recruiter receives part of these receipts, and a cut of future payments as the new recruits go on to recruit others. In reality, because of the geometrical progression of (hypothetical) recruits, few participants in a pyramid scheme will be able to recruit enough others to recover their initial investment, let alone make a profit, because the pool of potential recruits is rapidly exhausted.

Although they are illegal in many countries, pyramid schemes have existed for over a century. As recently as November 2008, riots broke out in several towns in Colombia after the collapse of several pyramid schemes, and in 2006 Ireland launched a website to better educate consumers to pyramid fraud after a series of schemes were perpetrated in Cork and Galway.

Perhaps the best-known type of pyramid scheme is a *chain letter*, which often does not involve even a fictitious product. A chain letter may contain k names; purchasers of the letter invest $\$2x$, with $\$x$ paid to the name at the top of the letter and $\$x$ to the seller of the letter. The purchaser deletes the name at the top of the list, adds his own at the bottom, and sells the letter to new recruits. The promoter's pitch is that if the purchaser, and each subsequent recruit for k-1 stages, sells just two letters, there will be 2^{k-1} people selling 2^k letters featuring the purchaser's name at the top of the list, so that the participant would net $\$2^k x$ from the venture. Many variants of this basic "get rich quick" scheme have been, and continue to be, promoted.

A structure that can be used to model many pyramid schemes is that of recursive trees. A tree with n vertices labeled $1, 2, \ldots, n$ is a *recursive tree* if node 1 is distinguished as the *root*, and for each j with $2 \leq j \leq n$, the labels of the vertices in the unique path from the root to node j form an increasing sequence. The special case of *random* or *uniform* recursive trees, in which all trees in the set of trees of given order n are equally probable, has been extensively analyzed (cf. [8], for example); however, most pyramid schemes or chain letters in practice have restrictions making their probability models non-uniform. The number of places where the next node may join the tree is then a random variable, unlike the uniform case. This complicates the analysis considerably (and may account for the relative sparsity of mathematical analysis of the properties of pyramid schemes).

Bhattacharya and Gastwirth (1983) analyze a chain letter scheme allowing reentry, in which each purchaser may sell only two letters, unless he purchases a new letter to re-enter the chain. In terms of recursive trees, this means that a node of the tree is *saturated* once it has two offspring nodes, and no further nodes can attach to it. It is further assumed that at each stage, participants who have not yet sold two letters all have an equal chance to make the next sale, i.e., all unsaturated nodes of the recursive tree have an equal chance of being the "parent" of the next node to be added. If L_n denotes the number of leaves of the recursive tree (nodes with no offspring) at stage n under this growth rule, L_n/n corresponds to the proportion of "shutouts" (those receiving no revenue) in this chain letter scheme. The analysis of Bhattacharya and Gastwirth sets up a nonhomogeneous Markov chain model and derives a diffusion approximation for large n. They find that L_n/n converges to 0.382 in this model and that the (centered and scaled) number of shutouts has a normally distributed limit. Mahmoud (1994) considers the height h_n of the tree of order n in this same "random pyramid" scheme and show that it converges with probability 1 to 3.98912; the proof involves embedding the discrete-time growth process of the pyramid in a continuous time birth-and-death process. Mahmoud notes that a similar analysis could be carried out for schemes permitting the sale of m letters, provided that the probabilistic behavior of the total number of shutouts could be derived (as it was in the binary case by Bhattacharya and Gastwirth).

Gastwirth (1977) and Gastwirth and Bhattacharya (1984) analyze another variant of pyramid schemes, known as a quota scheme. This places a limit on the maximum number of participants, so that the scheme corresponds to a recursive tree of some fixed size n. This scheme derives from a case in a Connecticut court (Naruk(1975)) in which people bought dealerships in a "Golden Book of Values", then were paid to recruit other dealers. In this scheme, each participant receives a commission from all of his descendants; thus for the j^{th} participant, the size of the branch of the tree rooted at j determines his profit. If S_j denotes the size of this branch, and j/n converges to a limit θ, Gastwirth and Bhattacharya showed that the distribution of

S_j converges to the geometric law

$$P(S_j = i + 1) = \theta(1 - \theta)^i \quad for \ i = 0, 1, 2, \ldots$$

(It was later shown (cf. [5]) that if j is fixed, the limiting distribution of S_j/n is $Beta(1, j-1)$.) Calculations made by Gastwirth and Bhattacharya show that, for example, when n is fixed at 270, the 135^{th} entry has probability only about 0.15 of recruiting two or more new entrants, and probability 0.03 of three or more recruits. Gastwirth (1977) shows that for large n, the expected proportion of all participants who are able to recruit at least r persons is 2^{-r}.

Other variants of pyramid schemes include the "8-Ball Model" and the "2-Up System" ([6]). In the 8-Ball model, the participant again recruits two new entrants, but does not receive any payment until two further levels have been successfully recruited. Thus a person at any level in the scheme would theoretically receive $2^3 = 8$ times his "participation fee", providing incentive to help those in lower levels succeed. In the 2-Up scheme, the income from a participant's first two recruits goes to the individual who recruited the participant; if the participant succeeds in recruiting three or more new entrants, the income received from these goes to the participant, along with the income from the first two sales made by each subsequent recruit. This scheme creates considerable incentive to pursue the potentially lucrative third recruit. For both of these schemes, it is easily calculated that when the pool of prospective recruits is exhausted, the majority of the participants in the scheme end up losing money.

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