Nicolaus BERNOULLI
b. 10 October 1687 - d. 29 November 1759

Summary. This member of the Bernoulli dynasty was, for a short period in the second decade of the eighteenth century, the leading figure in all of stochastics, and he has had a lasting influence. He edited his uncle's *Ars conjectandi* and this has often been mistakenly regarded as his sole contribution. He is sometimes confused with a cousin.

Nicolaus Bernoulli is from the second generation of mathematicians in the prominent family rooted in Basel, Switzerland. The tradition had been established by his two uncles, the famous brothers Jacob (1654–1705) (q.v.) and Johann (1667–1748), and the second generation is complete with his three younger cousins Nicolaus (1695–1726), Daniel (1700–1782) (q.v.) and Johann Jr. (1710–1790) – sons of Johann. Also known as Nikolaus, Niklaus, Nicolas or Nicholas (except for Johannes, we prefer the "neutral" Latin versions of given names that the Bernoullis themselves used on many of their publications), most histories of mathematics mix him with and mistakenly attribute some of his contributions to his cousin Nicolaus. While he is routinely given credit for the editing of the posthumous publication in 1713 of Jacob's *Ars conjectandi*, our Nicolaus' rôle in the development of these subjects was not recognized until very recently. O.B. Sheynin and A. Hald pointed out, in 1970 and 1984, the relevance of his work as a bridge between Jacob's law of large numbers and de Moivre's corresponding normal approximation, and subsequently Yushkevich [5] has made a very strong case for his importance in general. This would not have been possible without the scholarly publication of the book [1]. However, it was Hald [4] who analyzed Nicolaus Bernoulli out of obscurity, presenting him as an outstanding figure in the early development of probabilistic and statistical ideas.

This field as a mathematical discipline was new at the time, born in the correspondence between Pascal (q.v.) and Fermat (q.v.) in 1654 (the year when Jacob was born), and it was known from the sixteen pages of Huygens' (q.v.) *De ratiociniis in ludo aleae* of 1657. Pascal's own posthumous 1665 booklet, mainly on the combinatorics of the arithmetical triangle bearing his name, was little known at the time. As is clear from his *Meditationes* published in [1], Jacob Bernoulli took up studying the subject when he was about thirty years of age.

Nicolaus Bernoulli was born in 1687 in Basel. His father Nicolaus (1662–

1716), the second son of his grandfather Nicolaus (1623–1708) between Jacob and Johann, besides serving on the Basel city council, was an artist: the known portrait of Jacob, resting his right hand on a globe, is his painting. The portrait dates from the same year, 1687, when Jacob occupied the Chair of Mathematics at the University of Basel as professor. For the decades of our primary interest here, Basel was the leading mathematical center of Continental Europe. Significantly, 1687 was also the year of the publication of Newton's *Philosophiae naturalis principia mathematica.* This culmination of a scientific revolution also marked the beginning of a transition of everything mathematical. The Leibnizian version of the Newton – Leibniz differential and integral calculus was in fact championed by Jacob and Johann Bernoulli; as Leibniz wrote in 1694 to the brothers (just before the two engaged in vehement quarrels): "This method is no less yours than mine." *Analysis* (as it has become known in Continental Europe after l'Hôpital's influential text of 1696, practically bought in pieces from Johann) and the combinatorics of Leibniz and Jacob Bernoulli offered everything that was technically needed for what was becoming the *Art of Conjecturing* in Jacob's hands. It was perhaps even more important that the success of the *Principia*'s deductive approach for the description of the "system of the world" projected great hopes also for subjects of moral philosophy of the time. To simplify and exaggerate: Newton has discovered the design of and created order in the Heavens, so what was left was to do the same in human affairs on Earth. The mechanics of this feat was to be probability and, not paradoxically for such thinking, the philosophical or psychological basis for all this was the same Protestant determinism – on the part of Jacob Bernoulli – as for Newton for his piece. Even though the language of probability was necessarily that of games of chance for the most part at the time, the title of Part Four of Jacob's *Ars conjectandi* promises to apply the preceding doctrine "*in Civilibus, Moralibus, & Oeconomicis.*" The unpublished book stopped unfinished after the proof of the law of large numbers (the phrase dating from Poisson (q.v.) in 1837) for an unknown fraction or probability, illustrated by a numerical example, when its author, having suffered for some thirteen years from the fever of what was likely to have been tuberculosis, died in his fifty-first year.

Nicolaus successfully argued on parts of his uncle Jacob's work on infinite series for the title of a magister of arts in 1704. All the Bernoulli mathematicians before and after him had a comprehensive education, in theology, law or medicine, as well as in mathematics and natural philosophy. Having taken some more mathematical courses from uncle Johann after 1705, who then took over the direction of his studies, Nicolaus moved towards law. Trying to combine it with the *Ars* of Jacob, which he must have known from the manuscript of his original master, he defended his *dissertatio inauguralis mathematico-juridica* for a doctorate in law at the University of Basel in 1709, when he was still only 21, entitled *De usu artis conjectandi in jure.* Following Huygens' and Pascal's booklets from 1657 and 1665 and the extremely influential first edition of de Montmort's (q.v.) *Essay d'analyse sur les jeux de hazard* in 1708, the 56-page dissertation is the fourth bigger *published* work on probabilistic reasoning and, as an effort to try and fulfill at least part of Jacob's programme, certainly the very first on "applied" problems; its 12-page summary appeared in the *Acta Eruditorum* in 1711. (The original dissertation is reprinted in [1], accompanied by K. Kohli's set of commentaries.) Nicolaus was unaware of Pascal's book and from the dating of various correspondence it is also clear that de Montmort's complimentary copy of his 1708 *Essay* to Johann reached Basel only after Nicolaus defended his thesis in June, 1709.

Giving an overview of his uncle's ideas concerning the *Art of Conjecturing* in general, the dissertation discusses the calculation from grouped data of mean and median residual life-time functions, with a clear distinction between the two, given a survival or life-time distribution function (at least in points ten years apart), including joint life expectations; in particular, he determines the expected value of the largest of any given number of independent life-times, uniformly distributed on an interval. The results are motivated by many examples from civil and canon law and used for the estimation of the probability that an absent person is dead, for the determination of the purchase price of life annuities and the evaluation of bequests of maintenances, usufructs and life incomes, of life and marine insurance policies, and of the expected number of surviving children in the context of inheritance. Two problems he deals with are different in nature, but both are related to what will later become almost an obsession of the author: fairness. One is the study of fairness in lotteries, where Nicolaus advises the magistracy to allow profits only for charity and public good, the other is concerned with ideas about estimating the credibility of witnesses, suspicions and testimony in general. As Condorcet's (q.v.) extensive notes on Nicolaus Bernoulli's thesis ([2]) testify, the dissertation was rather influential even 75 years after its appearance. Through Condorcet's published work the last two problems later influenced Laplace (q.v.) and particularly Poisson (q.v.).

Except for the last two problems, things depend crucially on the underlying survival function. John Graunt's (q.v.) *Natural and Political Observations made upon the Bills of Mortality* (1662) contained the first life table, which, however, was *ad hoc* in most part. A 1686 paper of Jacob Bernoulli took over Graunt's life table from a 1666 resumé published in the *Journal des sçavans*. Believing that it was based on precise records of age at death, which it was not, this is what is used by nephew Nicolaus throughout his dissertation, though – unsatisfied with his own assumption of the uniform distribution of the lifetime in the ten-year periods of Graunt's life table – he mentions some Swiss data collected for him, but unfortunately gives only 18 mean residual life-times. Problems of this sort were discussed by the Huygens brothers, Jan de Witt, Jan Hudde, Edmond Halley and others before him, particularly the pricing of annuities, but, except for Halley's paper, these were not published.

Uncle Johann had his best praise about his nephew's talent and work in his letters to Leibniz. During Nicolaus' shorter trip to Paris in 1709 and his "grand tour" to France, England, The Netherlands and France again in 1712–1713, the doors of most mathematicians and scientists were open to him; among many others he met de Montmort (q.v.) in France, de Moivre (q.v.), Halley and Newton in London and Willem 'sGravesande in the Hague. Also, Johann introduced him to correspondence with several mathematicians. This is how Nicolaus began to exchange letters with de Montmort, which turned out to be very significant for our story: Johann, commenting on many problems in the 1708 *Essay* (pp. 283–298 in [3]), enclosed in his letter of March 17, 1710, Nicolaus' notes on de Montmort's famous matching problem (pp. 299–303 in [3]). Indeed, most of what we know about Nicolaus Bernoulli's work in stochastics besides his dissertation is from the seven letters of the published part of his correspondence with de Montmort up to 1713, included in the second edition of the *Essay* [3], pp. 299–412, from his correspondence with 'sGravesande in 1712, occupying 16 pages in the latter's *Oeuvres* published only in 1774 (these two sets are brilliantly analyzed by Hald [4]), from short excerpts of his later correspondence with de Montmort, Cramer and his cousin Daniel Bernoulli (q.v.), appearing in [1], and a few more letters of his correspondence with Leibniz and Euler published in other collections. At least 450 more letters to and from him, owned by the University of Basel, still remain unpublished.

The debut with de Montmort was spectacular. In February 1711, well within three months after receiving de Montmort's encouragement to do it, Nicolaus gave a complete solution to the latter's problem on the duration of play, "the most difficult topic in probability theory before 1750" according to Hald [4], which was approached less completely by de Moivre several times later. This is the determination of the probability that in the gambler's ruin problem (with both players having an arbitrary, possibly different number of ducats to play with and arbitrary winning probabilities p and q, $p+q=1$, at each game) the play ends with one of the players ruined in at most n games. The usual formulae for eventual ruin, stated without proof by Jacob in his *Ars* and proved in de Moivre's *De mensura sortis* in 1712, follow as $n \to \infty$.

The next problem was what an Englishman, a certain Mr. Waldegrave, posed to both de Montmort and de Moivre: players $\mathrm{P}_1, \ldots, \mathrm{P}_n$ of equal skill play a circular tournament, in which first P_1 and P_2 play, then the winner plays P_3, the winner of this match plays P_4, and so on, $\mathrm{P}_1, \ldots, \mathrm{P}_{n-1}$ entering again after P_n if necessary, until someone beats everyone else in a row. The problem is to calculate the probability of winning for each player, with the expected payoff if each loser pays a crown and the winner takes all, and the probability that the tournament ends on a given number of games. De Montmort and de Moivre could do this for $n = 3$ and $n = 4$, respectively, Nicolaus gave the solution in its complete generality also in 1711 and thought that this was his best contribution up to that point. This is the rare exception when the solution was also published, in the *Philosophical Transactions*, besides [3]: Nicolaus discussed it with de Moivre in 1712 while in London and subsequently sent a Latin version to him as well, and he arranged for its publication. Nicolaus and de Montmort then corresponded about numerous other games, such as the one named *Her*, which is historically the first example of solving a strategic game of chance.

Nicolaus' following contribution is what posterity probably takes as his best overall. It is in a letter again to de Montmort, dated on January 23, 1713 in Paris, on his way back from London (pp. 388–392 in [3], Russian and English translations in [5]; a version was also sent to 'sGravesande earlier). Improving on Jacob's proof for the law of large numbers for a relative frequency, he gave a large-sample approximation to a lower bound of the probability that a binomially distributed count of "fertile cases" does not deviate from its integer mean more than a given integer limit. The lower bound is expressed in terms of ratios of the middle and corresponding extreme terms of the binomial distribution and the approximation comes close to the local normal approximation established by de Moivre twenty years later. He was motivated by a statistical question: John Arbuthnot (q.v.)

extended Graunt's data on the christenings in London and used the excess of boys in all the 82 years 1629–1710 for his "Argument for Divine Providence ...," published in 1712 in the *Philosophical Transactions*, concluding that "it is Art, not Chance, that governs." This was a hot topic while Nicolaus was in London and he discussed it with 'sGravesande in the Hague, who improved Arbuthnot's argument for testing (and rejecting) $p = \frac{1}{2}$, also in 1712, for the probability p that a newborn is a boy. Using his approximate bound, Bernoulli, giving separate considerations to the middle and extreme portions of the data, fits a binomial distribution to the observed numbers and, taking $p = \frac{18}{35}$, concludes that "there is no ground to be surprised that the number of infants of the two sexes do not differ more [than observed], which I wanted to show." Misunderstood, as if he wanted to go against the theological conclusion of Arbuthnot, his findings were criticized as late as in the third edition of de Moivre's *Doctrine of Chances* in 1756, pp. 252–253. The kind of data analysis Nicolaus Bernoulli performed was not repeated until much later works by Daniel Bernoulli and Laplace.

From Paris Nicolaus went to de Montmort's country estate and stayed with him for about two months to help him prepare the second edition [3] of his *Essay*. (It would be a joint book by today's standards; both editions were published anonymously, but everyone knew who the author was.) He just got home to Basel, in April 1713, to write a preface for the publication of the *Ars conjectandi* and to include a list of the printer's errata, and it seems that was all he did, or was allowed to do by Jacob's widow and son, for the edition. (Neither the printer nor Jacob's son, arranging for the publication, was a mathematician. And to add to the Nicolaus confusion, this cousin of our Nicolaus, Jacob's son, and a painter, was also Nicolaus (1687–1769), born in the same year, and known as "Nicolaus the younger" to distinguish him from his uncle, our Nicolaus' father, "Nicolaus the elder.") The book came out in August 1713, three months before de Montmort's [3]; it could have been a point that this should happen so.

At this juncture, we see Nicolaus Bernoulli as *the* leading figure of the three main players (de Montmort, de Moivre and himself) in what Hald [4] describes as "the great leap forward" in stochastics from the publication of the first edition of de Montmort's book in 1708 (greatly influenced by the 1706 reviews of Jacob's unpublished book in the *Journal des sçavans*) to the first edition of de Moivre's book *Doctrine of Chances* in 1718. The latter was an extended and greatly improved English version of the 1712 *De mensura sortis*. Although de Moivre was years Nicolaus' senior by twenty years, and an accomplished mathematician by that time, he was a relatively late-comer to stochastics in 1712 (and already 66 years old when he found the normal approximation later in 1733). Nicolaus' help and leading rôle is fully acknowledged by de Montmort ([3], p. 400). He was elected to the Berlin Academy in May 1713 and to the Royal Society of London a year later. And, mysteriously enough, on this early zenith of his career the 26-year old leader of stochastics virtually quit his researches in his field of prominence.

He waited for three years in Basel for a job opportunity and, backed by Leibniz, became the professor of mathematics in Padua in 1716. A Swiss Protestant, probably not liking the Italian Catholicism surrounding him there, as Yushkevich [5] surmises, he returned to Basel in 1719, first occupied the chair of logic at the University of Basel in 1722 and then became the professor *utriusque iuris* (of both Roman and canon laws) in 1731, being well respected in that position to the end of his life. He did publish some good works in Italian journals and in the *Acta Eruditorum* on differential equations before 1721 and led a correspondence with Euler on infinite series in the years 1742–1745. He also tried to continue the correspondence with de Montmort after 1713 on some games of chance, but de Montmort, maybe bored by his admitted rôle of a second fiddler in this, was decreasingly interested and, before coming close to completing his next big project of writing a history of mathematics, died of smallpox in 1719. However, the enthusiasm and the quality of these activities of Bernoulli are on a much lower level than those of his great period of stochastics for the remaining 46 years of his life.

The real explanation for his leaving stochastics must come from the state of stochastics itself. As Nicolaus saw it, with the publication of the *Ars conjectandi* and the *Essay* in 1713 and de Moivre's *De mensura sortis* in 1712, what was left was really just Jacob's programme itself. There would have been small but inconvenient problems if someone wanted to give a try starting it. For example, Nicolaus no doubt learned in London that there was a better life table published by Halley in the *Philosophical Transactions* as early as 1694. Strictly speaking, his dissertation should have been rewritten with the moral that it was difficult to get quality data. (Had he known the first edition of de Montmort's *Essay* in 1709, he would have found a laudatory reference to Halley's paper.) The greater problem was that no significant concrete problems were coming forward *in Civilibus, Moralibus, & Oeconomicis*; and anyway, the statistical tools would have been missing even to treat them. Jacob's programme, or dream rather, was wholly impossible to accomplish in the eighteenth century. It is impossible today, and will remain so, in the sense that it was understood then. Indeed, it is almost comical how the programme appears in the prefaces of the books. De Montmort described this programme in his preface in 1708 from the eulogies of Jacob and concluded that he "will not have a Part Four" in his work and will "leave it to another person more capable than me." Nicolaus Bernoulli, in his preface to the *Ars conjectandi* wrote in 1713 that he was too young and inexperienced for the task of completing Jacob's Part Four and that, knowing that de Montmort would reprint his own preface in the second edition within a few months, he would rather ask de Montmort and de Moivre "to take the task on themselves." Finally, de Moivre threw the ball back in the preface of his *Doctrine of Chances* in 1718: "I wish I were capable of carrying on a Project [Jacob Bernoulli] had begun, of applying the Doctrine of Chances to *Oeconomical* and *Political* Uses, to which I have been invited, together with Mr. *de Montmort*, by Mr. *Nicholas Bernoully*: ... but I willingly resign my share of that task into better Hands, wishing that either he himself would prosecute that Design, ... or that his Uncle, Mr. *John Bernoully*, Brother to the Deceased, could be prevailed upon to bestow some of his Thoughts upon it."

So, Nicolaus' original idea of "carrying on the project" hardly appeared possible in 1713. To complicate matters further, he accidentally invented a problem in the same year (again in a letter to de Montmort; [3], pp. 401–402) that shook his and others confidence in what appeared to be the basic notion since Huygens: expectation. It was about Paul's fair price for the game in which Peter gives him $1, 2, 4, 8, 16, \dots$ ducats if he, flipping a fair coin, gets the first 'head' on the first, second, third, fourth, fifth, ... trial. The price is an infinite number of ducats according to Paul's infinite expectation, but "any even half-way sensible person would happily sell his chance for twenty ducats," as Nicolaus put it in 1728. Expectation to him was the mathematical version of justice and equity, the exact foundation on which fairness in human affairs could and would be built, and this is exactly what he managed to blow up. From here on, his work in stochastics is virtually restricted to correspondence on this paradox, known from 1768 as the St. Petersburg problem, trying to convince people that it was important and criticizing their work if he was successful. This is what happened in the case of Cramer and his cousin Daniel in the period from 1728 to 1732; the simplified statement of the problem above is due to Cramer, Nicolaus' original formulation was for throwing a die until the first 'six' appears. While the well-known utility approach of Cramer and Daniel Bernoulli has enjoyed an extremely distinguished career in economic theories later, their “moral expectations” never satisfied Nicolaus. He wanted to see a mathematics problem solved, but it was so difficult that it had to wait 242 years until stochastics got mature enough to afford it a reasonable initial treatment, a rough first approximation by a law of large numbers of Feller in 1945.

It is his early contributions in his great five years 1709–1713 to a reduced version of Jacob’s programme of stochastics, probability and statistics, for which Nicolaus Bernoulli is remembered today.

References

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