geometry, had achieved a strict axiomatic foundation. Geometry in the nineteenth century had flourished as never before, but it was chiefly in Hilbert's *Grundlagen* than an effort was first made to give it the purely formal character found in algebra and analysis. Euclid's *Elements* did have a deductive structure, to be sure, but it was replete with concealed assumptions, meaningless definitions, and logical inadequacies. Hilbert understood that not all terms in mathematics can be defined and therefore began his treatment of geometry with three undefined objects—point, line, and plane—and six undefined relations—being on, being in, being between, being congruent, being parallel, and being continuous. In place of Euclid's five axioms (or common notions) and five postulates, Hilbert formulated for his geometry a set of twenty-one assumptions, since known as Hilbert's axioms. Eight of these concern incidence and include Euclid's first postulate, four are on order properties, five are on congruency, three are on continuity (assumptions not explicitly mentioned by Euclid), and one is a parallel postulate essentially equivalent to Euclid's fifth postulate.[14] Following the pioneer work by Hilbert, alternative sets of axioms have been proposed by others; and the purely formal and deductive character of geometry, as well as other branches of mathematics, has been thoroughly established since the beginning of the twentieth century.

8 Hilbert, through his *Grundlagen*, became the leading exponent of an "axiomatic school" of thought which has been influential in fashioning contemporary attitudes in mathematics and mathematical education.[15] The *Grundlagen* opened with a motto taken from Kant: "All human knowledge begins with intuitions, proceeds to concepts, and terminates in ideas," but Hilbert's development of geometry established a decidedly anti-Kantian view of the subject. It emphasized that the undefined terms in geometry should not be assumed to have any properties beyond those indicated in the axioms. The intuitive-empirical level of the older geometrical views must be disregarded, and points, lines, and planes are to be understood merely as elements of certain given sets. Set theory, having taken over algebra and analysis, now was invading geometry. Similarly, the undefined relations are to be treated as abstractions indicating nothing more than a correspondence or mapping. Through analytic geometry the formal treatment of geometry had been associated with the axiomatization of algebra, and the ultimate outcome of the association was a degree of abstraction exceeding anything

[14] A list of the postulates can be found in Ralph G. Stanton and Kenneth D. Fryer, *Topics in Modern Mathematics* (Englewood Cliffs, N.J.: Prentice-Hall, 1964), pp. 167–170, as well as in the various editions of Hilbert's *Foundations*.

[15] See, for example, Rolf Nevanlinna, "Reform in Teaching Mathematics," *The American Mathematical Monthly*, **73** (1966), 451–464.