Andrei Andreevich MARKOV
b. 2 June 1856 (o.s.) - d. 20 July 1922

Summary. Markov, with Liapunov a disciple of Chebyshev, gave rigorous proofs of the Central Limit Theorem. Through his work on Markov chains, the concept of Markovian dependence pervades modern theory and application of random processes. His textbook influenced the development of probability and statistics internationally.

Markov was born in Ryazan, and died in Petrograd (which was - before the revolution, and is now again - called St. Petersburg). He was a poor student in all but mathematics at the fifth Petersburg gymnasium which he entered in 1866. Already during this period he revealed an emotional and uncompromising nature which was to surface in clashes with the tsarist regime and academic colleagues, even though his motives were generally high-minded. He was, however, more fortunate in his circumstances than his similarly volatile younger countryman E.E. Slutsky (q.v.) in that Markov had influential senior colleagues who understood and tolerated him, among whom V.A. Steklov is mentioned frequently; and in that he worked in the capital city of the Russian Empire.

Entering Petersburg University in 1874, he attended classes in the Physico-Mathematical Faculty by A.N. Korkin, E.I. Zolotarev and P.L. Chebyshev (q.v.), all of whom encouraged him and facilitated his progress. At the completion of his studies in 1878 he received a gold medal and was retained by the university to prepare for a career as an academic, in the tradition of the times with the best students.

With the departure of Chebyshev from the university in 1883, Markov took over his course in probability which he continued to teach yearly. Markov's doctoral dissertation *On Some Applications of Algebraic Continued Fractions*, results from which were published in 1884, already had implicit connections with probability theory inasmuch as it treated certain inequalities published by Chebyshev in 1874 in Liouville's *J. Math. Pures Appl.*, relating to the method of moments which Chebyshev had in turn extracted from notions of I.J. Bienaymé (q.v.). At the proposal of Chebyshev, Markov was elected to the St. Petersburg Academy of Science in 1886, attaining full membership in 1896. With A.M. Liapunov, Markov became the most eminent of Chebyshev's disciples in probability of the Petersburg "School", and remained closest to his teacher's ideas. The writings of Markov and Liapunov placed probability on the level of an exact mathematical science. Markov's published probabilistic work has in much of its background correspondence with A.V. Vasiliev, professor at Kazan University, a graduate of the same Petersburg gymnasium, and also a student of Chebyshev. Indeed several important papers of Markov, including the one in which "Markov chains" first appear in Markov's writings in 1906, were published in the *Izvestiia (Bulletin)* of the Physico-Mathematical Society of Kazan University.

The initial impetus for Markov's work in probability theory was Chebyshev's proof, which was incomplete, of the Central Limit Theorem. Bouncing his ideas off Vasiliev, Markov begins in 1898 by replacing one of Chebyshev's conditions, while persisting with Chebyshev's approach via the method of moments. Liapunov's Theorem on the Central Limit problem, published in 1901, differs not only in its approach (by characteristic functions, originating with Cauchy and I.V. Sleshinsky (1854-1931)) but also in its level of generality. This motivated Markov to wonder whether the method of moments might not be suitably adapted to give the same result; and he finally achieved this in 1913 in the 3rd edition of his *Ischislenie Veroiatnostei (Calculus of Probabilites)*.

Markov was embroiled in several controversies with the Moscow mathematician P.A. Nekrasov (1853-1924), one of which led (Seneta, 1984, 1996) to Markov's outstanding contribution to probability theory, the concept of chain dependence of random variables. The first of these controversies was initiated by a probabilistic paper of Nekrasov in 1898, dedicated to Chebyshev (!) and containing no proofs. It was followed by about 1,000 pages of obscure and verbose argument in *Matematicheskii Sbornik*. In its attempt to establish now-standard local and global theorems of Central Limit type for large deviations, this work of Nekrasov was ahead of its time, but was only partly successful. Its specific inaccuracies were criticised by Markov and Liapunov, who never understood the general direction; the task of so doing was formidable. Moreover, in the course of these writings in 1902 Nekrasov claimed that pairwise independence of summands was a necessary condition for the Weak Law of Large Numbers (WLLN) to hold. He had examined, he said, the "logical underpinnings" of the way the Bienaymé-Chebyshev Inequality was used to prove the WLLN. The *observed* stability of averages in everyday life, through the claimed consequent necessity of pairwise independence, justified the doctrine of free will. It was this attempt to use mathematics and statistics in support of theological doctrine which led Markov to construct a scheme of *dependent* random variables in his Kazan paper, which ends, without ever mentioning Nekrasov explicitly, with the words

> "Thus, independence of quantities does not constitute a necessary condition for existence of the law of large numbers."

It was in 1902 also that Markov protested over the reversal by the tsar of election as Honorary Member of the Academy of Science of A.M. Gorky (Peshkov). Markov refused subsequently to accept any awards ("orders") from the Academy, or to act as "agent of the government" in relation to students at the university. He came into conflict with the Council of Petersburg University in 1905 about the procedure for relaxing the quota on admission of Jews. In 1912 when the Synod of the Russian Orthodox Church excommunicated Leo Tolstoy, Markov likewise requested excommunication. His character and beliefs in combination with his scientific eminence were very acceptable to the incoming political system following the October revolution in 1917, and contributed in having the Petersburg School put into exclusive eminence in Soviet mathematical historiography (in contrast to the Moscow School, of which Nekrasov and later Egorov and Luzin were members).

The same historiographic tendencies have progressively ascribed the Bienaymé-Chebyshev Inequality and the method of moments to Chebyshev alone. But just as Chebyshev in the 1874 paper had given Bienaymé due credit, so Markov too was ever a defender of Bienaymé's priority. In response to a statement of Nekrasov that the idea of Bienaymé is exhausted in the works of Chebyshev who, Nekrasov continues, himself had remarked on this in 1874, Markov in 1912 writes (characteristically)

> "The reference here to Chebyshev is misleading, and the statement of P.A. Nekrasov that the idea of Bienaymé is exhausted is contradicted by a sequence of my papers containing a generalization of the method of Bienaymé to settings which are not even touched on in the writings of P.A. Nekrasov."

The first of these papers which he lists is the Kazan paper of 1906, written to contradict Nekrasov's assertions about the necessity of pairwise independence for the WLLN.

Markov retired from the university in 1905, but continued to teach probability theory there. From 1904 to 1915 he wrote letters to newspapers on current social issues, and especially on education (Sheynin, 1989); the press coined for him the name *Neistovy Andrei (Andrew the Furious).* In 1915 he opposed the programme proposed by P.S. Florov and Markov's continuing bête noir Nekrasov about changes to the school mathematics syllabus. There are good biographies of Markov, most notably by his son (Markov, 1951) and Grodzensky (1987).

It is, however, his views of and contributions to statistics which deserve to be addressed also.

On his retirement from the university, continuing to seek practical applications of probability theory, he participated from the beginning in deliberations on running the retirement fund of the Ministry of Justice, following in the footsteps of his probabilistic predecessors V. Ya. Buniakovsky, M.V. Ostrogradsky and Bienaymé.

Markov's attention was turned to mathematical statistics through his correspondence (Ondar, 1981) with Chuprov (q.v.) which begins 2 November 1910 with a postcard to the latter criticizing him for mentioning Nekrasov's name in the same breath as Chebyshev's, in Chuprov's erudite *Ocherki po Teorii Statistiki (Topics in the Theory of Statistics)* of 1909, which had just come to Markov's attention. From such inauspicious beginnings, in which Markov, claiming to judge all work only from a strictly mathematical point of view, dismissed the work of Karl Pearson (q.v.) amongst others, grew a lively correspondence on the topic of dispersion theory. At the same time as the interests of the statistician Chuprov were turned progressively to a mathematical direction, Markov's negative attitude to statistics softened, and in the end, out of the correspondence came elegant and important theoretical contributions from both (Heyde and Seneta, 1977, Section 3.4). Indeed, the correspondence marks the coming together in the Russian Empire of probability and statistics into mathematical statistics. The correspondence ends in early 1917. In the course of it, Markov was led in 1913 to modelling the alternation of vowels and consonants in several Russian literary works by a two-state Markov chain and estimation in the model using dispersion-theoretic ideas.

Markov was also interested, through the influence of Chebyshev, in the classical linear model which he treated in his *Ischislenie Veroiatnostei* in various editions. The inappropriate name "Gauss-Markov theorem" seems ultimately to arise from these treatments.

Markov's Inequality is the name given to the result $P(Y \geq a) \leq EY/a$ where Y is a non-negative random variable and $a > 0$. It appeared in the 1913 edition of Markov's *Ischislenie Veroiatnostei*, and is more fundamental than the Bienaymé-Chebyshev Inequality, although the simple proof used by

Bienaymé can be modified to prove it also.

References

[1] Grodzensky, S. Ya. (1987). *Andrei Andreevich Markov. 1856-1922.* Nauka, Moscow.

[2] Heyde, C.C. and Seneta, E. (1977). *I.J. Bienaymé. Statistical Theory Anticipated* Springer, New York.

[3] Markov, A.A. (1951). *Izbrannie Trudy. (Selected Works).* ANSSSR, Leningrad.

[4] Ondar, Kh.O. (1981). *The Correspondence Between A.A. Markov and A.A. Chuprov on the Theory of Probability and Mathematical Statistics.* Springer, New York [Transl. by C. and M. Stein].

[5] Seneta, E. (1984). The central limit problem and linear least squares in pre-revolutionary Russia. The background. *Mathematical Scientist*, **9**, 37-77.

[6] Seneta, E. (1996). Markov and the birth of chain dependence theory. *International Statistical Review*, **64**, 255-263.

[7] Sheynin, O.B. (1989). A.A. Markov's work on probability. *Archive for History of Exact Sciences*, **39**, 337-377.

E. Seneta