THE FIVE GROUPS OF AXIOMS.

§ 1. THE ELEMENTS OF GEOMETRY AND THE FIVE GROUPS OF AXIOMS.

Let us consider three distinct systems of things. The things composing the first system, we will call *points* and designate them by the letters A, B, C,...; those of the second, we will call *straight lines* and designate them by the letters a, b, c,...; and those of the third system, we will call *planes* and designate them by the Greek letters α, β, γ,... The points are called the *elements of linear geometry*; the points and straight lines, the *elements of plane geometry*; and the points, lines, and planes, the *elements of the geometry of space* or the *elements of space.*

We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated," "between," "parallel," "congruent," "continuous," etc. The complete and exact description of these relations follows as a consequence of the *axioms of geometry.* These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition. We will name these groups as follows:

I,	1–7.	Axioms of *connection.*
II,	1–5.	Axioms of *order.*
III.		Axiom of *parallels* (Euclid's axiom).
IV,	1–6.	Axioms of *congruence.*
V.		Axiom of *continuity* (Archimedes's axiom).

§ 2. GROUP I: AXIOMS OF CONNECTION.

The axioms of this group establish a connection between the concepts indicated above; namely, points, straight lines, and planes. These axioms are as follows:

I, 1. *Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$.*

Instead of "determine," we may also employ other forms of expression; for example, we may say A "lies upon" a, A "is a point of" a, a "goes through" A "and through" B, a "joins" A "and" or "with" B, etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression: "The straight lines" a "and" b "have the point A in common," etc.

I, 2. *Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then is also $BC = a$.*

I, 3. *Three points A, B, C not situated in the same straight line always completely determine a plane α. We write $ABC = a$.*

Theorem 20. The sum of the angles of a triangle is two right angles.

Definitions. If M is an arbitrary point in the plane α, the totality of all points A, for which the segments MA are congruent to one another, is called *a circle*. M is called the *centre of the circle*.

From this definition can be easily deduced, with the help of the axioms of groups III and IV, the known properties of the circle; in particular, the possibility of constructing a circle through any three points not lying in a straight line, as also the congruence of all angles inscribed in the same segment of a circle, and the theorem relating to the angles of an inscribed quadrilateral.

§ 8. GROUP V. AXIOM OF CONTINUITY. (ARCHIMEDEAN AXIOM.)

This axiom makes possible the introduction into geometry of the idea of continuity. In order to state this axiom, we must first establish a convention concerning the equality of two segments. For this purpose, we can either base our idea of equality upon the axioms relating to the congruence of segments and define as "*equal*" the correspondingly congruent segments, or upon the basis of groups I and II, we may determine how, by suitable constructions (see Chap. V, § 24), a segment is to be laid off from a point of a given straight line so that a new, definite segment is obtained "*equal*" to it. In conformity with such a convention, the axiom of Archimedes may be stated as follows:

V. Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Take the points $A_2, A_3, A_4, \ldots$ so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4 etc. Moreover, let the segments

$$AA_1,\ A_1A_2,\ A_2A_3,\ A_3A_4,\ \ldots$$

be equal to one another. Then, among this series of points, there always exists a certain point A_n such that B lies between A and A_n.

The axiom of Archimedes is a *linear* axiom.

Remark.[3] To the preceeding five groups of axioms, we may add the following one, which, although not of a purely geometrical nature, merits particular attention from a theoretical point of view. It may be expressed in the following form:

Axiom of Completeness.[4] *(Vollständigkeit): To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.*

[3]Added by Professor Hilbert in the French translation.—*Tr.*

[4]See Hilbert, "Ueber den Zahlenbegriff," *Berichte der deutschen Mathematiker-Vereinigung*, 1900.

This axiom gives us nothing directly concerning the existence of limiting points, or of the idea of convergence. Nevertheless, it enables us to demonstrate Bolzano's theorem by virtue of which, for all sets of points situated upon a straight line between two definite points of the same line, there exists necessarily a point of condensation, that is to say, a limiting point. From a theoretical point of view, the value of this axiom is that it leads indirectly to the introduction of limiting points, and, hence, renders it possible to establish a one-to-one correspondence between the points of a segment and the system of real numbers. However, in what is to follow, no use will be made of the "axiom of completeness."