Blaise PASCAL
b. 19 June 1623 - d. 19 August 1662

Summary Pascal introduced the concept of mathematical expectation and used it recursively to obtain a solution to the Problem of Points which was the catalyst that enabled probability theory to develop beyond mere combinatorial enumeration.

Blaise Pascal was born in Clermont, France. In 1631 his father Etienne Pascal (himself an able mathematician who gave his name to the '*limaçon* of Pascal') moved his family to Paris in order to secure his son a better education, and in 1635 was one of the founders of Marin Mersenne's 'Academy', the finest exchange of mathematical information in Europe at the time. To this informal academy he introduced his son at the age of fourteen, and Blaise immediately put his new source of knowledge to good use, producing (at the age of sixteen) his *Essay pour les coniques*, a single printed sheet enunciating Pascal's Theorem, that the opposite sides of a hexagon inscribed in a conic intersect in three collinear points.

At the age of eighteen the younger Pascal turned his attention to constructing a calculating machine (to help his father in his calculations) and within a few years he had built and sold fifty of them. Some still exist. In 1646 he started work on hydrostatics, determining the weight of air experimentally and writing on the vacuum (leading ultimately to the choice of 'Pascal' as the name for the S.I. unit of pressure).

In 1654 Pascal returned to mathematics, extending his early work on conics in a manuscript which was never printed and which does not now exist, though it was seen by Leibniz. In the same year he entered into correspondence with Pierre de Fermat (q.v.) of Toulouse about some problems in calculating the odds in games of chance which led him to write the *Traité du triangle arithmétique, avec quelques autres petits traitez sur la mesme matière*, probably in August of that year. Not published until 1665, this work, and the correspondence itself which was published in 1679, is the basis of Pascal's reputation in probability theory as the originator of the concept of expectation and its use recursively to solve the 'Problem of Points', as well as the justification for calling the arithmetical triangle 'Pascal's triangle'. His advances, considered to be the foundation of modern probability theory, are described in detail below.

Later in 1654 Pascal underwent a religious experience as a result of which he almost entirely abandoned his scientific work and devoted his remaining years to writing his *Lettres provinciales*, written in 1656 and 1657 in defence of Antoine Arnauld, and his *Pensées*, drafts for an *Apologie de la Religion Chrétienne*, which include his famous 'wager' (see below). In 1658-59 Pascal briefly returned to mathematics, writing on the curve known as the cycloid, but his final input into the development of probability theory arises through his presumed contribution to *La logique, ou l'art de penser* by Antoine Arnauld and Pierre Nicole, published in 1662. This classic philosophical treatise is often called the *Port-Royal Logic* through the association of its authors, and Pascal himself, with the Port-Royal Abbey, a centre of the Jansenist movement within the Catholic church, into whose convent Pascal's sister Jacqueline had been received in 1652.

Pascal died in Paris, in good standing with the Church and is buried in the church of St Etienne du Mont.

The *Traité du triangle arithmétique* itself is 36 pages long (setting aside *quelques autres petits traitez sur la mesme matière*) and consists of two parts. The first carries the title by which the whole is usually known, in English translation *A Treatise on the Arithmetical Triangle*, and is an account of the arithmetical triangle as a piece of pure mathematics. The second part *Uses of the Arithmetical Triangle* consists of four sections: (1) *Use in the theory of figurate numbers*, (2) *Use in the theory of combinations*, (3) *Use in dividing the stakes in games of chance*, (4) *Use in finding the powers of binomial expressions.*

Pascal opens the first part by defining an unbounded rectangular array like a matrix in which 'The number in each cell is equal to that in the preceding cell in in the same row', and he considers the special case in which the cells of the first row and column each contain 1. Symbolically, he has defined $\{f_{i,j}\}$ where

$$f_{i,j} = f_{i-1,j} + f_{i,j-1}, i, j = 2, 3, 4, ..., f_{i,1} = f_{1,j} = 1, i, j = 1, 2, 3, ...$$

The rest of Part I is devoted to a demonstration of nineteen corollaries flowing from this definition, and concludes with a 'problem'. The corollaries include all the common relations among the *binomial coefficients* (as the entries of the triangle are now universally called), none of which was new. Pascal proves the twelfth corollary, $(j-1)f_{i,j} = if_{i+1,j-1}$ in our notation, by explicit use of mathematical induction. The 'problem' is to find $f_{i,j}$ as a function of i and j, which Pascal does by applying the twelfth corollary recursively. Part I of the *Treatise* thus amounts to a systematic development of all the main results then known about the properties of the numbers in the arithmetical triangle.

In Part II Pascal turns to the applications of these numbers. The numbers thus defined have three different interpretations, each of great antiquity (to which he does not, however, refer). The successive rows of the triangle define the *figurate numbers* which have their roots in Pythagorean arithmetic. Pascal treats these in section (1),

The second interpretation is as *binomial numbers*, the coefficients of a binomial expansion, which are arrayed in the successive diagonals, their identity with the figurate numbers having been recognized in Persia and China in the eleventh century and in Europe in the sixteenth. The above definition of $f_{i,j}$ is obvious on considering the expansion of both sides of

$$(x+y)^n = (x+y)(x+y)^{n-1}.$$

The fact that the coefficient of $x^r y^{n-r}$ in the expansion of $(x+y)^n$ may be expressed as

$$\frac{n(n-1)(n-2)...(n-r+1)}{1.2.3...r} = \binom{n}{r}$$

was known to the Arabs in the thirteenth century and to the Renaissance mathematician Cardano in 1570. It provides a closed form for $f_{i,j}$, with $n = i + j - 2$ and $r = i - 1$. Pascal treats the binomial interpretation in section (4).

The third interpretation is as a *combinatorial number*, for the number of combinations of n different things taken r at a time, ${}^n\mathrm{C}_r$ is equal to $\binom{n}{r}$, a result known in India in the ninth century, to Hebrew writers in the fourteenth century, and to Cardano in 1550. Pascal deals with that interpretation in section (2), giving a novel demonstration of the combinatorial version of the basic addition relation ${}^{n+1}C_{r+1} = {}^n\mathrm{C}_r + {}^nC_{r+1}$, for, considering any particular one of the $n+1$ things, ${}^n\mathrm{C}_r$ gives the number of combinations that include it and ${}^nC_{r+1}$ the number that exclude it.

In section (3) Pascal breaks new ground, and this section, taken together with his correspondence with Fermat, is the basis of his reputation as the father of probability theory. The "problem of points" which he discusses, also known simply as the "division problem", involves determining how the total stake should be divided in the event of a game of chance being terminated prematurely. Suppose two players X and Y stake equal money on being the first to win n points in a game in which the winner of each point is decided by the toss of a fair coin. If such a game is interrupted when X still lacks x points and Y lacks y, how should the total stake be divided between them? In the middle of the sixteenth century Tartaglia famously concluded "the resolution of such a question is judicial rather than mathematical, so that in whatever way the division is made there will be cause for litigation". A century later, in the summer of 1654, the correct solution was derived by three different arguments during the correspondence between Pascal and Fermat. One of the methods, advanced by Pascal in his letter of July 29, involves computing expectations recursively.

The key development was his understanding that the value of a gamble is equal to its mathematical expectation computed as the average of the values of each of two equally-probable outcomes, and that this precise definition of value lends itself to recursive computation because the value of a gamble that one is certain to win is undoubtedly the total stake itself. Thus if the probabilities of winning a or b units are each one half, the expectation is $(a + b)$ units, which is then the value of the gamble. In *Ars conjectandi* (1713) James Bernoulli (q.v.) called this "the fundamental principle of the whole art". Pascal has invented the concept of "expected value", that is the probability of a win multiplied by its value, and understood that it is an exact mathematical concept which can be manipulated.

In the letter to Fermat, Pascal develops the recursive argument applied to expected values in order to find the correct division of the stake-money, and thus computes the "value" of each successive throw. In the *Traité* the same idea is more formally expressed, and in particular Pascal gives as a principle the value of the expectation when the chances are equal. He remarks at one point that "the division has to be proportional to the chances", but in the solution to the problem when the players have equal chances the question of computing an expectation for unequal chances does not arise; that extension was first formally made by Huygens (q.v.) in his *De ratiociniis in ludo aleae* of 1657. It is known that when Huygens spent July-September 1655 in Paris he had the opportunity to discuss Pascal's work on probability problems with Roberval, and presumably learnt of the concept of mathematical expectation then.

The easiest way to understand how Pascal used expectation and recursion to solve the problem of points is to visualise the event-tree of possible further results. Each bifurcation corresponds to a toss, one branch for X winning it and the other for Y, and successive bifurcations must lead eventually to tips corresponding to the whole game being won by either X or Y. Considering now the expectation of X (say), each tip can be labelled with his expectation, either S (the total stake) or 0, as the case may be. Applying now Pascal's expectation rule for equal chances, each bifurcation can have an expectation associated with it, working recursively down the tree from the tips to the root, at which point we find the solution to the problem. If $E(x, y)$ be the expectation of player X when he lacks x points and Y lacks y, then the recursion is

$$E(x, y) = E(x - 1, y) + E(x, y - 1).$$

By these methods, and his knowledge of the arithmetical triangle, Pascal was able to demonstrate how the stake should be divided between the players according to the partial sums of the binomial coefficients, a result which he had already obtained by enumeration, for both he and Fermat had realised that the actual game of uncertain length could be embedded in a game of fixed length to which the binomial distribution (as yet undiscovered) could then be applied. Further details of their reasoning may be found in the references below.

Three further contributions of Pascal to the development of probability theory may be noted. In *Pensées* he used his concept of expectation to argue that one should bet on the existence of God because, however small the probability of His existence, the value of eternal salvation if He does exist is infinite, so that the expected value of assuming that He does exist far exceeds that of assuming that He does not. Subsequent writers have regarded this, "Pascal's wager", as the origin of decision theory. Secondly, in 1656 Pascal posed Fermat a problem which became known as the "gambler's ruin" and which played a central role in the development of probability theory as the first example of a problem about the duration of play. Finally, he greatly influenced the probability arguments in the *Port-Royal Logic*, in the last chapter of which there is a clear understanding of the importance of judging an action not only by the possible gain or loss, but also by the probability of each of these. One must "consider mathematically the magnitudes when these things are multiplied together". This is the first occasion in which the word " probability" is used (in French, of course) in its modern sense.

References

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